THE WEAK EXTENSION PROPERTY AND FINITE AXIOMATIZABILITY FOR QUASIVARIETIES

WIESLAW DZIOBIAK, MIKLÓS MARÓTI, RALPH MCKENZIE, AND ANVAR NURAKUNOV

Abstract

We define and compare a selection of congruence properties of quasivarieties, including the relative congruence meet semi-distributivity, RSD($\wedge$), and the weak extension property, WEP. We prove that if $\mathcal{K} \subseteq \mathcal{L} \subseteq \mathcal{L}'$ are quasivarieties of finite signature, and $\mathcal{L}'$ is finitely generated while $\mathcal{K} \models$ WEP, then $\mathcal{K}$ is finitely axiomatizable relative to $\mathcal{L}$. We prove for any quasivariety $\mathcal{K}$ that $\mathcal{K} \models$ RSD($\wedge$) iff $\mathcal{K}$ has pseudo-complemented congruence lattices and $\mathcal{K} \models$ WEP. Applying these results and other results proved in M. Maróti, R. McKenzie [14] we prove that a finitely generated quasivariety of finite signature $\mathcal{L}$ is finitely axiomatizable provided that $\mathcal{L}$ satisfies RSD($\wedge$), or that $\mathcal{L}$ is relatively congruence modular and is included in a residually small congruence modular variety. This yields as a corollary the full version of R. Willard's theorem for quasivarieties and partially proves a conjecture of D. Pigozzi. Finally, we provide a quasi-Maltsev type characterization for RSD($\wedge$) quasivarieties and supply an algorithm for recognizing when the quasivariety generated by a finite set of finite algebras satisfies RSD($\wedge$).

1. Introduction

We interpret the class-operators $\mathsf{H}, \mathsf{S}, \mathsf{P}$ in the inclusive sense, so that for example, $\mathsf{S}(\mathcal{K})$ denotes the class of all algebras isomorphic to a subalgebra of some algebra in $\mathcal{K}$. A *quasivariety* is a class $\mathcal{K}$ of algebras of the same signature and such that $\mathsf{S}(\mathcal{K}) = \mathsf{P}(\mathcal{K}) = \mathsf{P}_u(\mathcal{K}) = \mathcal{K}$ where $\mathsf{P}_u(\mathcal{K})$ denotes the class of all algebras isomorphic to an ultraproduct of a system of algebras from $\mathcal{K}$. Since $\mathsf{P}(\mathcal{K})$ contains a product of an empty system of algebras, a quasivariety contains the one-element algebras of its signature. A *variety* is a quasivariety which, in addition, satisfies $\mathsf{H}(\mathcal{K}) = \mathcal{K}$. A quasivariety $\mathcal{K}$ is said to be *finitely generated* if $\mathcal{K} = \mathsf{SP}(\mathcal{M})$ for some finite set $\mathcal{M}$ of finite algebras. Likewise, a variety $\mathcal{K}$ is said to be *finitely generated* (i.e., as a variety) if $\mathcal{K} = \mathsf{HSP}(\mathcal{M})$ for some finite set $\mathcal{M}$ of finite algebras.

For a quasivariety $\mathcal{K}$ and $\mathbf{A} \in \mathcal{K}$ we write $\mathbf{Con\,A}$ for the lattice of congruences of $\mathbf{A}$ and $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$ for the lattice of $\mathcal{K}$-congruences of $\mathbf{A}$, where a

$\mathcal{K}$-congruence is any congruence θ such that $\mathbf{A}/\theta \in \mathcal{K}$. Recall that the join $\vee$ in $\mathbf{Con}\,\mathbf{A}$ and the join, $\vee^{\mathcal{K}}$, in $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$ are related by the equation

$$\alpha \vee^{\mathcal{K}} \beta = \overline{\alpha \vee \beta}^{\mathcal{K}}, \quad \text{for } \{\alpha, \beta\} \subseteq \mathrm{Con}_{\mathcal{K}}\mathbf{A}$$

where for any $\theta \in \mathrm{Con}\,\mathbf{A}$, $\overline{\theta}^{\mathcal{K}}$ is the smallest member of $\mathrm{Con}_{\mathcal{K}}\mathbf{A}$ containing θ.

This paper is devoted to the study of a collection of sufficient conditions for finite axiomatizability of a finitely generated quasivariety. All such conditions we encounter will be expressed as combinations of properties of the lattice of congruences, or the lattice of relative congruences (or a combination of the two) holding for all algebras in a quasivariety. We now define the congruence properties we shall use. Recall that a lattice $\mathbf{L}$ is said to be *distributive* (respectively *modular*, or *meet semi-distributive*) if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (respectively $\{x \geq y\} \rightarrow \{x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)\}$, or $\{x \wedge y = x \wedge z\} \rightarrow \{x \wedge (y \vee z) = x \wedge y\}$) holds for all elements x, y, z in $\mathbf{L}$.

A quasivariety is said to satisfy CM (respectively, CD or SD($\wedge$)) if and only if all the lattices $\mathbf{Con}\,\mathbf{A}$ with $\mathbf{A} \in \mathcal{K}$ are modular lattices (respectively, distributive lattices, or meet semi-distributive lattices). A quasivariety is said to satisfy RCM (respectively, RCD or RSD($\wedge$)) if and only if all the lattices $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$ with $\mathbf{A} \in \mathcal{K}$ are modular lattices (respectively, distributive lattices, or meet semi-distributive lattices).

Let $\mathcal{K}$ be any quasivariety. We say that $\mathcal{K}$ satisfies the *extension property*, EP, if and only if for all $\mathbf{A} \in \mathcal{K}$ the map $\theta \mapsto \overline{\theta}^{\mathcal{K}}$ is a lattice homomorphism of $\mathbf{Con}\,\mathbf{A}$ onto $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$. We say that $\mathcal{K}$ satisfies the *weak extension property*, WEP, if and only if for all $\mathbf{A} \in \mathcal{K}$ and $\alpha, \beta \in \mathrm{Con}\,\mathbf{A}$, if $\alpha \wedge \beta = 0_A$ then $\overline{\alpha}^{\mathcal{K}} \wedge \overline{\beta}^{\mathcal{K}} = 0_A$. We say that $\mathcal{K}$ satisfies the PCC, or has *pseudo-complemented congruences*, if for all $\mathbf{A} \in \mathcal{K}$ and $\alpha \in \mathrm{Con}\,\mathbf{A}$, there is a largest congruence $\beta \in \mathrm{Con}\,\mathbf{A}$ satisfying $\alpha \wedge \beta = 0_A$ (i.e., β is the pseudo-complement of α in $\mathbf{Con}\,\mathbf{A}$). We also write $\mathcal{K} \models \mathrm{P}$ to denote that $\mathcal{K}$ satisfies P. For example, $\mathcal{K} \models \mathrm{RSD}(\wedge)$ means that $\mathcal{K}$ satisfies RSD($\wedge$).

The serious study of the finite axiomatizability property in finitely generated varieties originated with Roger Lyndon in the 1950's. A decade earlier, Garrett Birkhoff had asked whether the equational theory of a finite algebra of finite signature must be finitely axiomatizable. Lyndon proved in [12] that the equational theory of every two-element algebra of finite signature is finitely axiomatizable and also provided in [13] the first example of a finite algebra (a 7-element groupoid) the equational theory of which is not finitely axiomatizable. In the half-century since these papers appeared, many interesting and important results concerning the finite or non-finite axiomatizability of finitely generated varieties have been established. (See [14] for a brief account of the results obtained). The following three results have been regarded with respect to the range of their applicability as the most distinguished positive results in this literature.

Baker's theorem (K. Baker [2]). *Let $\mathcal{K}$ be a variety of finite signature. If $\mathcal{K}$ is congruence distributive and finitely generated, then $\mathcal{K}$ is finitely axiomatizable.*

McKenzie's theorem (R. McKenzie [16]). *Let $\mathcal{K}$ be a variety of finite signature. If $\mathcal{K}$ is congruence modular, residually small, and finitely generated, then $\mathcal{K}$ is finitely axiomatizable.*

Willard's theorem (R. Willard [23]). *Let $\mathcal{K}$ be a variety of finite signature. If $\mathcal{K}$ is congruence meet semi-distributive and has a finite residual bound, then $\mathcal{K}$ is finitely axiomatizable.*

Once each of the above theorems had been proved, it prompted attempts to find its valid extensions within finitely generated quasivarieties, or equivalently speaking, within quasi-equational classes generated by finite sets of finite algebras (see for example W.J. Blok and D. Pigozzi [4], J. Czelakowski, W. Dziobiak [5], W. Dziobiak [6], K. Kearnes, R. McKenzie [11], M. Maróti and R. McKenzie [14], A. Nurakunov [20], and D. Pigozzi [21]). A quasi-equational class is a universal Horn class which contains a one-element algebra.

Extending Willard's theorem into quasivarieties M. Maróti, R. McKenzie [14] proved that if $\mathcal{K}$ is a finitely generated quasivariety of finite signature which has both PCC and WEP, then $\mathcal{K}$ is finitely axiomatizable.

We prove that $\mathrm{RSD}(\wedge) \Leftrightarrow \mathrm{PCC} + \mathrm{WEP}$. Thus we obtain the full version of Willard's theorem for quasivarieties.

In [21], D. Pigozzi proved the following theorem: If $\mathcal{K}$ is a finitely generated and relatively congruence distributive quasivariety of finite signature, then $\mathcal{K}$ is finitely axiomatizable. D. Pigozzi's theorem fully extended K. Baker's theorem into quasivarieties, and is itself extended by the new results mentioned above. Having proved his theorem and looked for a common extension into quasivarieties of both his and R. McKenzie's theorem, D. Pigozzi conjectured in the late 1980's the following: If $\mathcal{K}$ is a finitely generated and relatively congruence modular quasivariety of finite signature, then $\mathcal{K}$ is finitely axiomatizable.

K. Kearnes, R. McKenzie [11] developed a commutator theory for relative congruences in relatively congruence modular quasivarieties, with the expectation that it can be used to prove D. Pigozzi's conjecture; but the truth of this conjecture remains an open question.

We prove that D. Pigozzi's conjecture is true provided that $\mathcal{K}$ is additionally assumed to be included in a residually small congruence modular variety.

In view of R. Mckenzie's negative solution to A. Tarski's problem (see R. McKenzie [17]) which asked if there is an algorithm to determine whether the variety generated by a finite set of finite algebras of finite signature is finitely axiomatizable, and the fact that the analogue of A. Tarski's problem for quasivarieties has not been solved yet, theorems like those of K. Baker, R. McKenzie, D. Pigozzi, and R. Willard have been gaining an extra

meaning. All known theorems of this type provide sufficient conditions for a finitely generated variety or quasivariety of finite signature to be finitely axiomatizable. All the known sufficient conditions have been shown to be algorithmically verifiable. Continuing this tradition, we offer here, in Section 6 an algorihm for determining if a finite set of finite algebras generates a quasivariety satisfying RSD($\wedge$).

2. Implications between congruence properties

The congruence properties CM, CD, SD($\wedge$), RCM, RCD, RSD($\wedge$), EP, WEP and PCC were defined in the introduction. We require also two properties of a different character.

Let $\mathcal{K}$ be any quasivariety. A finite set of pairs of terms,

$$W = \{(s_i(x,y,z), t_i(x,y,z)) : 0 \leq i < n\},$$

is said to be a set of *Willard terms* for $\mathcal{K}$ if and only if the equations $s_i(x,y,x) \approx t_i(x,y,x)$ $(0 \leq i < n)$ are valid in $\mathcal{K}$ and

$$\mathcal{K} \models (\forall x, y) [x \neq y \rightarrow \bigvee_{i<n} (s_i(x,x,y) = t_i(x,x,y) \leftrightarrow s_i(x,y,y) \neq t_i(x,y,y))].$$

We write $\mathcal{K} \models W$ to denote that $\mathcal{K}$ has a set of Willard terms.

A finite set of pairs of terms

$$D = \{(p_i(x,y,z,u), q_i(x,y,z,u)) : 0 \leq i < n\}$$

is said to be a set of *quasi-Day terms* for $\mathcal{K}$ if and only if the equations

$$p_i(x,y,y,x) \approx q_i(x,y,y,x) \quad \text{and} \quad p_i(x,x,y,y) \approx q_i(x,x,y,y), \ 0 \leq i < n$$

are valid in $\mathcal{K}$ and

$$\mathcal{K} \models (\forall x, y, z) [x \neq z \rightarrow \bigvee_{i<n} p_i(x,y,y,z) \neq q_i(x,y,y,z)].$$

Most of the facts stated in the next theorem can be found either in K. Kearnes, R. McKenzie [11] or in M. Maróti, R. McKenzie [14]. We believe that Theorem 1(4) is new. The results in the corollary may be partly new.

Theorem 1. *For any quasivariety $\mathcal{K}$ the following hold.*
  (1) $CD \Leftrightarrow CM + SD(\wedge)$
  (2) $RCD \Leftrightarrow RCM + RSD(\wedge)$
  (3) $PCC \Leftrightarrow W$ and $SD(\wedge) \Rightarrow PCC$
  (4) $RSD(\wedge) \Leftrightarrow PCC + WEP \Leftrightarrow W + WEP$
  (5) $RCM \Rightarrow EP \Rightarrow WEP$
  (6) $RCM \Leftrightarrow \mathcal{K}$ has quasi-Day terms and satisfies WEP

Corollary 2. *Let $\mathcal{K} \subseteq \mathcal{L}$ be quasivarieties.*
  (i) *If $\mathcal{L}$ is RCM then $\mathcal{K}$ is RCM iff $\mathcal{K}$ satisfies WEP.*
  (ii) *If $\mathcal{L}$ is RSD($\wedge$) then $\mathcal{K}$ is RSD($\wedge$) iff $\mathcal{K}$ satisfies WEP.*
  (iii) *If $\mathcal{L}$ is RCM and $\mathcal{K}$ is RSD($\wedge$) then $\mathcal{K}$ is RCD.*
  (iv) *If $\mathcal{L}$ is RSD($\wedge$) and $\mathcal{K}$ is RCM then $\mathcal{K}$ is RCD.*

We remark that a quasivariety $\mathcal{K}$ is CM iff the variety $\mathcal{L} = \mathsf{H}(\mathcal{K})$ is CM and that here, $\mathcal{L} \models \mathrm{CM}$ iff $\mathcal{L} \models \mathrm{RCM}$. The analogous statements hold for the properties CD and $\mathrm{SD}(\wedge)$. Consequently, each of the four assertions of the corollary remains true when the hypothesis $\mathcal{L} \models \mathrm{RCM}$ or $\mathcal{L} \models \mathrm{RSD}(\wedge)$ is replaced by $\mathcal{K} \models \mathrm{CM}$ or, respectively, $\mathcal{K} \models \mathrm{SD}(\wedge)$.

Proof. (1) and (2): It is a well-known fact that any lattice is distributive iff it is both modular and meet (or join) semi-distributive.

(3): The part $\mathrm{PCC} \Leftrightarrow \mathrm{W}$ was proved in M. Maróti and R. McKenzie [14] (see Theorem 2.3 of [14]). The part $\mathrm{SD}(\wedge) \Rightarrow \mathrm{PCC}$ is an easy consequence of the fact that every algebraic meet semi-distributive lattice is pseudo-complemented.

(4): We prove $\mathrm{RSD}(\wedge) \Leftrightarrow \mathrm{PCC} + \mathrm{WEP}$ and obtain $\mathrm{RSD}(\wedge) \Leftrightarrow \mathrm{W} + \mathrm{WEP}$ as a consequence of (3).

Suppose that $\mathcal{K} \models \mathrm{PCC+WEP}$. Let $\mathbf{A} \in \mathcal{K}$, $\alpha, \beta, \gamma \in \mathbf{Con}_{\mathcal{K}}\mathbf{A}$, and $\alpha \wedge \beta = \alpha \wedge \gamma = 0_A$. By PCC, $\alpha \wedge (\beta \vee \gamma) = 0_A$ which, by $\alpha \in \mathbf{Con}_{\mathcal{K}}\mathbf{A}$ and WEP, gives $\alpha \wedge (\beta \vee^{\mathcal{K}} \gamma) = 0_A$. Thus $\mathcal{K} \models \mathrm{RSD}(\wedge)$

Next, we suppose that $\mathcal{K}$ satisfies $\mathrm{RSD}(\wedge)$ and we show that $\mathcal{K}$ has Willard terms which, by (3), will give $\mathcal{K} \models \mathrm{PCC}$. Let $\mathbf{F} = \mathbf{F}_{\mathcal{K}}(3)$ be the free algebra in $\mathcal{K}$ freely generated by x, y, z. Let $\alpha = \mathrm{Cg}_{\mathbf{F}}(x, z)$, the congruence generated by the pair (x, z); and let $\beta = \mathrm{Cg}_{\mathbf{F}}(x, y)$ and $\gamma = \mathrm{Cg}_{\mathbf{F}}(y, z)$. Notice that each of α, β, γ is the kernel of an endomorphism of $\mathbf{F}$, and hence $\{\alpha, \beta, \gamma\} \subseteq \mathrm{Con}_{\mathcal{K}}\mathbf{F}$. Put

$$
\begin{aligned}
\beta_0 &= \beta \\
\gamma_0 &= \gamma \\
\beta_{n+1} &= \beta \vee^{\mathcal{K}} (\alpha \wedge \gamma_n) \\
\gamma_{n+1} &= \gamma \vee^{\mathcal{K}} (\alpha \wedge \beta_n).
\end{aligned}
$$

This defines increasing sequences $\langle \beta_n : n \in \omega \rangle$, $\langle \gamma_n : n \in \omega \rangle$ in $\mathbf{Con}_{\mathcal{K}}\mathbf{F}$. Then $\beta_\infty = \bigvee_n \beta_n$, $\gamma_\infty = \bigvee_n \gamma_n$ are $\mathcal{K}$-congruences. By join-continuity, $\alpha \wedge \beta_\infty = \alpha \wedge \gamma_\infty$. Now since $\mathcal{K} \models \mathrm{RSD}(\wedge)$, then $\alpha \wedge \beta_\infty = \alpha \wedge (\beta_\infty \vee^{\mathcal{K}} \gamma_\infty)$. Thus

$$
\alpha \wedge (\beta \vee \gamma) \leq \alpha \wedge (\beta_\infty \vee \gamma_\infty) \leq \alpha \wedge \beta_\infty,
$$

showing that $(x, z) \in \beta_\infty$.

Now write

$$
M = \{(s(x, y, z), t(x, y, z) \in F^2 : \mathcal{K} \models s(x, y, x) \approx t(x, y, x)\}.
$$

In fact, $M = \alpha$. Suppose that $\mathbf{A} \in \mathcal{K}$, $a, b \in A$, and

$$
\mathbf{A} \models s(a, a, b) = t(a, a, b) \leftrightarrow s(a, b, b) = t(a, b, b) \quad \text{for all } (s, t) \in M.
$$

This means that where f_0, f_1 are the homomorphisms $\mathbf{F} \to \mathbf{A}$ with

$$
(f_0(x), f_0(y), f_0(z)) = (a, a, b), \quad (f_1(x), f_1(y), f_1(z)) = (a, b, b),
$$

and $\beta' = \ker(f_0)$, $\gamma' = \ker(f_1)$, we have $\alpha \wedge \beta' = \alpha \wedge \gamma'$. Notice that $\beta', \gamma' \in \mathrm{Con}_{\mathcal{K}}\mathbf{F}$ and $\beta \leq \beta'$, $\gamma \leq \gamma'$. An easy calculation then shows inductively that

$$\beta_n \leq \beta', \ \gamma_n \leq \gamma', \ \text{for all } n.$$

We conclude that $(x, z) \in \beta'$ (since $(x, z) \in \beta_\infty$). But this just means that $a = b$ (i.e., $f_0(x) = f_0(z)$).

We have shown that

$$\mathcal{K} \models (\forall x, z) [x \neq z \rightarrow \bigvee_{(s,t)\in M} (s(x, x, z) = (x, x, z) \leftrightarrow s(x, z, z) \neq t(x, z, z))].$$

With a logical compactness argument, we deduce that the displayed statement remains true when M is replaced by some finite subset W of M. Clearly, W is a set of Willard terms for $\mathcal{K}$.

This concludes our proof that $\mathrm{RSD}(\wedge) \Rightarrow \mathrm{W}$.

Finally, we prove that $\mathrm{RSD}(\wedge) \Rightarrow \mathrm{WEP}$. It will be a proof by contradiction. Suppose that $\mathcal{K} \models \mathrm{RSD}(\wedge)$, and that $\mathbf{A} \in \mathcal{K}$, $\{\theta, \psi\} \subseteq \mathrm{Con}\,\mathbf{A}$, $\theta \wedge \psi = 0_A$, and $\overline{\theta}^{\mathcal{K}} \wedge \overline{\psi}^{\mathcal{K}} > 0_A$. If $\theta \wedge \overline{\psi}^{\mathcal{K}} = 0_A$, replace θ by $\overline{\psi}^{\mathcal{K}}$, and replace ψ by θ. Thus we can assume that $\theta \wedge \psi = 0_A$, $\theta \wedge \overline{\psi}^{\mathcal{K}} > 0_A$.

Now let

$$B = \{(x, y) \in A^2 : \ \text{there exist } c, d \in A \text{ so that } x\theta c\psi y \ \text{and} \ x\psi d\theta y\}.$$

That is to say, $B = (\theta \circ \psi) \cap (\psi \circ \theta)$. B is of course a subuniverse of $\mathbf{A} \times \mathbf{A}$ that includes $\theta \cup \psi$ and we write $\mathbf{B}$ for the corresponding algebra. Where $(x, y) \in B$ and c, d are as above, the elements c, d are unique. We have four surjective homomorphisms $\mathbf{B} \rightarrow \mathbf{A}$:

$$\begin{aligned}
\pi_0(x, y) &= x; \\
\pi_1(x, y) &= y; \\
\varphi(x, y) &= c \ \text{ where } x\theta c\psi y; \\
\varphi'(x, y) &= d \ \text{ where } x\psi d\theta y.
\end{aligned}$$

The kernels of these homomorphisms are, of course, $\mathcal{K}$-congruences of $\mathbf{B}$. They are

$$\begin{aligned}
0_0 &= \ker(\pi_0); \\
0_1 &= \ker(\pi_1); \\
\theta_0 \wedge \psi_1 &= \ker(\varphi); \\
\psi_0 \wedge \theta_1 &= \ker(\varphi').
\end{aligned}$$

Thus

$$\begin{aligned}
\alpha &= \ker(\varphi) \wedge \ker(\pi_1) = \theta_0 \wedge 0_1, \\
\beta &= \ker(\varphi') = \psi_0 \wedge \theta_1, \\
\gamma &= \ker(\pi_0) \wedge \ker(\varphi) = 0_0 \wedge \psi_1
\end{aligned}$$

are $\mathcal{K}$-congruences of $\mathbf{B}$.

Obviously (since $\theta \wedge \psi = 0_A$) we have $\alpha \wedge \beta = \alpha \wedge \gamma = 0_B$. Since $\mathcal{K} \models \mathrm{RSD}(\wedge)$, then $\alpha \wedge \overline{\beta \vee \gamma}^{\mathcal{K}} = 0_B$. We can calculate $\beta \vee \gamma$ as

$$\beta \vee \gamma = \psi_0\,.$$

To see this, note first that $\beta \cup \gamma \subseteq \psi_0$. Next, suppose that $((x_0,y_0),(x_1,y_1)) \in \psi_0$. This means simply that $(x_0, x_1) \in \psi$. We can calculate:

$$(x_0,y_0)\gamma(x_0,\varphi(x_0,y_0))\beta(x_1,x_0)\gamma(x_1,x_1)\beta(x_1,\varphi(x_1,y_1))\gamma(x_1,y_1)\,.$$

Next, note that from the isomorphism $\mathbf{B}/\psi_0 \cong \mathbf{A}/\psi$, it follows that the interval $1_A/\psi$ in $\mathbf{Con\,A}$ is naturally isomorphic with the interval $1_B/\psi_0$ in $\mathbf{Con\,B}$ via the map $\tau \mapsto \tau_0$, where for all such τ, $\mathbf{A}/\tau \cong \mathbf{B}/\tau_0$. From this, we deduce that $(\overline{\psi}^{\mathcal{K}})_0 = \overline{\psi_0}^{\mathcal{K}}$. Now

$$\alpha \wedge \overline{\psi_0}^{\mathcal{K}} = \alpha \wedge \overline{\beta \vee \gamma}^{\mathcal{K}} = 0_B\,,$$

since $\mathcal{K} \models \mathrm{RSD}(\wedge)$. But on the other hand, we have some $(x,y) \in \theta \wedge \overline{\psi}^{\mathcal{K}}$, $x \neq y$. Then $(x,y),(y,y) \in B$ and obviously,

$$((x,y),(y,y)) \in \alpha \wedge \overline{\psi_0}^{\mathcal{K}}\,,$$

which contradicts the displayed equations just above. This ends our proof that $\mathrm{RSD}(\wedge) \Rightarrow \mathrm{WEP}$.

Our proof of (4) is now complete. Statement (5) is Theorem 1.1 in K. Kearnes, R. McKenzie [11]. Theorem 4.1 in the same paper demonstrates that RCM is equivalent to the conjunction of EP and the existence of quasi-Day terms. But in the paragraph immediately following the conclusion of their proof of Theorem 4.1, those authors point out that our stronger Theorem 1 (6) is true and show how to prove it.

To prove the corollary, assume that $\mathcal{K} \subseteq \mathcal{L}$ are quasivarieties. It is clear that if $\mathcal{L}$ satisfies PCC (equivalently, W), or has quasi-Day terms, then $\mathcal{K}$ has the same property. Thus (i) and (ii) follow immediately from Theorem 1 (4) and (6). Statements (iii) and (iv) then follow from the equivalence $\mathrm{RCD} \Leftrightarrow \mathrm{RCM} + \mathrm{RSD}(\wedge)$. Since we did not prove Theorem 1 (6), and the proof given in K. Kearnes, R. McKenzie [11] is rather involved, we shall now give a direct proof of the fact that if $\mathcal{L}$ is relatively congruence modular and $\mathcal{K}$ has the weak extension property, then $\mathcal{K}$ is relatively congruence modular.

Assume that $\mathcal{L}$ has RCM and $\mathcal{K}$ has WEP. To see that $\mathcal{K}$ has RCM, let $\mathbf{A} \in \mathcal{K}$ and $\alpha, \beta, \gamma \in \mathrm{Con}_{\mathcal{K}}\mathbf{A}$ with $\alpha \geq \beta$. We must prove that $\alpha \wedge (\beta \vee^{\mathcal{K}} \gamma) = \beta \vee^{\mathcal{K}} (\alpha \wedge \gamma)$. By replacing β with $\beta \vee^{\mathcal{K}} (\alpha \wedge \gamma)$, we can assume that $\alpha \wedge \gamma \leq \beta$. Next, replacing $\mathbf{A}$ by $\mathbf{A}/(\alpha \wedge \gamma)$ (an algebra in $\mathcal{K}$) and replacing α by $\alpha/(\alpha \wedge \gamma)$, β by $\beta/(\alpha \wedge \gamma)$, γ by $\gamma/(\alpha \wedge \gamma)$, we can assume that $\alpha \wedge \gamma = 0_A$. Note that $\{\alpha, \beta, \gamma\} \subseteq \mathrm{Con}_{\mathcal{L}}\mathbf{A}$ and we do have $\alpha \wedge (\beta \vee^{\mathcal{L}} \gamma) = \beta \vee^{\mathcal{L}} (\alpha \wedge \gamma) = \beta$. Now α, β are in $\mathrm{Con}_{\mathcal{K}}\mathbf{A}$ and where $\tau = \beta \vee^{\mathcal{L}} \gamma$, obviously $\overline{\tau}^{\mathcal{K}} = \beta \vee^{\mathcal{K}} \gamma$. Applying the WEP in the algebra $\mathbf{A}/\beta$, it follows from $\alpha \wedge \tau = \beta$ that $\alpha \wedge \overline{\tau}^{\mathcal{K}} = \beta$—i.e., $\alpha \wedge (\beta \vee^{\mathcal{K}} \gamma) = \beta = \beta \vee^{\mathcal{K}} (\alpha \wedge \gamma)$, as desired.

We recommend to the reader the proof, in K. Kearnes, R. McKenzie [11] (Theorem 1.1), that RCM $\Rightarrow$ WEP (and in fact implies EP). It is a nice argument. $\qquad\square$

3. Quasi-Maltsev type characterization of RSD($\wedge$)

As in K. Kearnes and R. McKenzie [11], by a *quasi-Maltsev condition* we mean a condition expressed as the disjunction of conditions each of which postulates the existence of a finite set of terms satisfying certain quasi-equations. In general, any quasi-Maltsev condition alone cannot characterize RSD($\wedge$). The reason is that a RSD($\wedge$) quasivariety may have a subquasi-variety that is not RSD($\wedge$). However, RSD($\wedge$) can be characterized by the conjunction of a quasi-Maltsev condition and the weak extension property, as shown by Theorem 3 below. A similar characterization for RCM quasi-varieties was given in K. Kearnes and R. McKenzie [11]; see Theorem 1(6) in this paper.

Let $\mathcal{K}$ be a quasivariety and $\mathbf{F} = \mathbf{F}_{\mathcal{K}}(3)$ be the free algebra in $\mathcal{K}$ freely generated by x, y, z. Let $\alpha = \mathrm{Cg}_{\mathbf{F}}(x, z)$, $\beta = \mathrm{Cg}_{\mathbf{F}}(x, y)$, and $\gamma = \mathrm{Cg}_{\mathbf{F}}(y, z)$. In the proof of Theorem 1, we defined the following two sequences of congruences.

$$
\begin{aligned}
\beta_0 &= \beta \\
\gamma_0 &= \gamma \\
\beta_n &= \beta \vee^{\mathcal{K}} (\alpha \wedge \gamma_{n-1}) \\
\gamma_n &= \gamma \vee^{\mathcal{K}} (\alpha \wedge \beta_{n-1}) \,.
\end{aligned}
$$

We will refer to these congruences again; see Theorem 3 and its proof given below.

A *tree* is a finite ordered set $(T, \leq)$ having a smallest element, called the *root* of T, and the property that every set of the form $r \downarrow = \{x \in T : x \leq r\}$ is a chain in T. If $t < r$ and $r \downarrow = \{r\} \cup t \downarrow$, then r is called a *child* of t. A *leaf* of T is the same thing as a maximal element in $(T, \leq)$. For $t \in T$, we set $h(t) = |t \downarrow| - 1$ and $C_T(t) = \{r \in T : r \text{ is a child of } t\}$. We also set $h(T) = \max \{h(t) : t \in T\}$. A tree is said to be *non-trivial* if $h(T) > 0$.

Theorem 3. *For any quasivariety $\mathcal{K}$ the following conditions are equivalent:*
   (i) *$\mathcal{K}$ is RSD($\wedge$);*
   (ii) *$\mathcal{K}$ satisfies WEP and $(x, z) \in \beta_n$ for some $n \in \omega \setminus \{0\}$;*
   (iii) *$\mathcal{K}$ satisfies WEP and there exists a finite set $\Delta = \{(p_t, q_t) : t \in T\} \subseteq F^2$ where the index set T of Δ has the structure of a non-trivial tree such that, for every $t \in T$, the following equations and quasi-equations are universally valid in $\mathcal{K}$:*

   (1) *If t is the root of T, then $p_t(x, y, z) \approx x$ and $q_t(x, y, z) \approx z$;*
   (2) *If t is not a leaf in T, then $p_t(x, y, x) \approx q_t(x, y, x)$ and*
   $\bigwedge\{p_r(x, y, z) \approx q_r(x, y, z) : r \in C_T(t)\} \rightarrow p_t(x, y, z) \approx q_t(x, y, z);$

(3) *If $0 < h(t) < h(T)$ and t is a leaf in T, then $p_t(x, x, y) \approx q_t(x, x, y)$ when $h(t)$ is odd, and $p_t(x, y, y) \approx q_t(x, y, y)$ when $h(t)$ is even;*

(4) *If $h(t) = h(T)$, then either $p_t(x, x, y) \approx q_t(x, x, y)$, or $p_t(x, y, x) \approx q_t(x, y, x)$ and $p_t(x, y, y) \approx q_t(x, y, y)$ when $h(T)$ is odd, and either $p_t(x, y, y) \approx q_t(x, y, y)$, or $p_t(x, y, x) \approx q_t(x, y, x)$ and $p_t(x, x, y) \approx q_t(x, x, y)$ when $h(T)$ is even.*

Proof. (i) $\Rightarrow$ (ii): It follows from (i) that $\alpha \wedge (\beta \vee \gamma) \leq \beta_\infty$ – see the proof of Theorem 1. This gives $(x, z) \in \beta_n$ for some $n \in \omega \setminus \{0\}$.

(ii) $\Rightarrow$ (iii): Assume that $\mathcal{K}$ satisfies WEP, $n > 0$, and $(x, z) \in \beta_n$. The idea of the proof of this implication is simple. We begin with the following claim, which is a consequence of the definitions of β_k and γ_k and a compactness argument.

<u>Claim 1:</u> For $k > 0$ and 3-ary terms p and q, if $(p, q) \in \beta_k$ (or $(p, q) \in \gamma_k$, respectively) then there are finite sets $A \subseteq \beta$ and $B \subseteq \alpha \cap \gamma_{k-1}$ (or $A \subseteq \gamma$ and $B \subseteq \alpha \cap \beta_{k-1}$, respectively), such that $\mathcal{K} \models \bigwedge\{r \approx s : (r, s) \in A \cup B\} \Rightarrow p \approx q$.

Now for every odd integer i with $0 \leq i < n$, choose (using the claim) a mapping $\sigma \mapsto (A_i(\sigma), B_i(\sigma))$ defined on γ_{n-i} so that for $\sigma = (p, q) \in \gamma_{n-i}$, $A_i(\sigma)$ is a finite subset of γ, $B_i(\sigma)$ is a finite subset of $\alpha \cap \beta_{n-i-1}$, $A_i(\sigma) \cap \alpha \cap \beta_{n-i-1} = \emptyset$, and

$$\mathcal{K} \models \bigwedge\{r \approx s : (r, s) \in A_i(\sigma) \cup B_i(\sigma)\} \to p \approx q.$$

By placing $(x, x) \in B_i(\sigma) \setminus A_i(\sigma)$, we can assume that $B_i(\sigma) \neq \emptyset$. For every even integer i with $0 \leq i < n$, choose a mapping $\sigma \mapsto (A_i(\sigma), B_i(\sigma))$ defined on β_{n-i} so that for $\sigma = (p, q) \in \beta_{n-i}$, $A_i(\sigma)$ is a finite subset of β, $B_i(\sigma)$ is a finite non-empty subset of $\alpha \cap \gamma_{n-i-1}$, $A_i(\sigma) \cap \alpha \cap \gamma_{n-i-1} = \emptyset$, and

$$\mathcal{K} \models \bigwedge\{r \approx s : (r, s) \in A_i(\sigma) \cup B_i(\sigma)\} \to p \approx q.$$

The labeled tree T we shall use to satisfy (iii) is now determined. We construct it as follows.

For the elements of T we take all finite sequences $t = (\sigma_0, \ldots, \sigma_i)$ of elements of F^2 satisfying: $0 \leq i \leq n$; $\sigma_0 = (x, z)$; for $0 \leq j < i$, $\sigma_j \in \beta_{n-j}$ if j is even, $\sigma_j \in \gamma_{n-j}$ if j is odd, and $\sigma_j \in \alpha$ if $0 < j < i$; for $0 \leq j < i$, $\sigma_{j+1} \in A_j(\sigma_j) \cup B_j(\sigma_j)$. (Note that these conditions imply that for $0 \leq j < i - 1$ then $\sigma_{j+1} \in B_j(\sigma_j)$, but $\sigma_i \in A_{i-1}(\sigma_{i-1})$ is allowed when $A_{i-1}(\sigma_{i-1}) \neq \emptyset$.)

T is ordered by putting $t \leq t'$ iff t is an initial segment of t'. This obviously makes T into a rooted tree with root $\perp = ((x, z))$. For $t = (\sigma_0, \ldots, \sigma_i) \in T$ we put $p_t(x, y, z) = p$, $q_t(x, y, z) = q$ where $\sigma_i = (p, q)$. The tree T has height n. We leave to the reader the straightforward proof that T with the labels (p_t, q_t), $t \in T$, satisfies (iii).

(iii) $\Rightarrow$ (i): Assume (iii). In particular, assume that $\mathcal{K}$ satisfies WEP and that we have a non-trivial tree T with labels (p_t, q_t), $t \in T$, that satisfies

(1)–(4) of statement (iii). Since $\mathcal{K} \models \text{WEP}$, in order to prove (i) it is enough, by Theorem 1(4), to show that $\mathcal{K}$ has Willard terms.

$\underline{\text{Claim 2:}}$ $W = \{(p_t, q_t) : t \in T \text{ and } \mathcal{K} \models p_t(x, y, x) \approx q_t(x, y, x)\}$ is a set of Willard terms for $\mathcal{K}$.

To prove this claim, let $\mathbf{A} \in \mathcal{K}$, $\{a, b\} \subseteq A$, and suppose that for all $(p_t, q_t) \in W$ we have

$$p_t(a, a, b) = q_t(a, a, b) \leftrightarrow p_t(a, b, b) = q_t(a, b, b).$$

We must show that $a = b$.

Where $n = h(T)$, let I be the set of all i, $0 \leq i \leq n$, such that: (I) if i is odd and $i = h(t)$, then $p_t(a, a, b) = q_t(a, a, b)$; and (II) if i is even and $i = h(t)$, then $p_t(a, b, b) = p_t(a, b, b)$. We show that $I = \{0, 1, \ldots, n\}$, by reverse induction on i, starting at $i = n$ and working down to $i = 0$. As a result we will have that $0 \in I$, implying that, where $\bot$ is the root of T, $a = p_\bot(a, b, b) = q_\bot(a, b, b) = b$, as desired.

For the base step of the induction, consider a vertex t of height n. If n is odd, and $p_t(x, x, y) \approx q_t(x, x, y)$ is not a valid equation of $\mathcal{K}$, then by (4) of the condition (iii) of Theorem 3, $(p_t, q_t) \in W$ and $p_t(a, b, b) = q_t(a, b, b)$. In this case, $p_t(a, a, b) = q_t(a, a, b)$ due to our assumption about a, b. On the other hand, if $p_t(x, x, y) \approx q_t(x, x, y)$ is an equation of $\mathcal{K}$, then of course $p_t(a, a, b) = q_t(a, a, b)$. This shows that $n \in I$ if n is odd. The argument for the case of even n is analogous, and also uses (4) of the condition (iii) of Theorem 3.

Now assume that $0 \leq i < n$ and $i + 1 \in I$. We show that $i \in I$. So let t be a vertex of T of height i. Let say, i be even. First, assume that t is not a leaf. Now for every child r of t in T, $h(r) = i + 1$ and so $p_r(a, a, b) = q_r(a, a, b)$ by the inductive assumption. Then, by (2) of the condition (iii) of Theorem 3, we have $p_t(a, a, b) = q_t(a, a, b)$, and also we have $(p_t, q_t) \in W$. Thus it follows that $p_t(a, b, b) = q_t(a, b, b)$, as required. The case where i is odd and t is not a leaf yields to a similar argument.

Finally, assume that t is a leaf of height i. Then by (3) of (iii) of Theorem 3, we have that $p_t(a, a, b) = q_t(a, a, b)$ if i is odd, and if i is even, then $p_t(a, b, b) = q_t(a, b, b)$. This concludes our proof that $i + 1 \in I$ implies $i \in I$; and that concludes our proof of Theorem 3. □

Corollary 4. *A quasivariety $\mathcal{K}$ has Willard terms if and only if it admits a labelled tree satisfying condition (iii) of Theorem 3. If T, with labels (p_t, q_t), $t \in T$, is such a tree for $\mathcal{K}$, then $W = \{(p_t, q_t) : t \in T \text{ and } \mathcal{K} \models p_t(x, y, x) \approx q_t(x, y, x)\}$ is a set of Willard terms for $\mathcal{K}$.*

Proof. Claim 2 in the proof of Theorem 3 shows that if T is such a tree for $\mathcal{K}$ then W is a set of Willard terms for $\mathcal{K}$. Conversely, it is proved in M. Maroti, R. McKenzie [14] that a quasivariety has Willard terms iff it has pseudo-complemented congruence lattices, and consequently, the possession of Willard terms implies that $(x, z) \in \beta_n$ for some $n > 0$. □

The tree condition (iii) in Theorem 3 is similar to the condition shown by R. Willard [22] to be equivalent to congruence meet semi-distributivity for a variety, and from which he extracted the existence of what we have called Willard terms.

4. Finite Axiomatizability

Our purpose in this section is to prove a very general result about finite axiomatizability (Theorem 6), and use it to prove that every finitely generated quasivariety satisfying RSD($\wedge$), and having finite signature, is finitely axiomatizable (Theorem 7). Using Theorem 4, we also partially prove D. Pigozzi's conjecture (Theorem 8).

Let us recall from the introduction some notable finite axiomatizability theorems. Suppose that $\mathcal{K}$ is a finitely generated quasivariety of finite signature. K. Baker [2] proved that $\mathcal{K}$ is finitely axiomatizable, provided $\mathcal{K}$ satisfies CD + $\mathsf{H}(\mathcal{K}) = \mathcal{K}$ (i.e., $\mathcal{K}$ is a congruence distributive variety). D. Pigozzi [21] proved more generally that $\mathcal{K}$ is finitely axiomatizable if $\mathcal{K}$ satisfies RCD. R. Willard [22] extended K. Baker's theorem in another direction: $\mathcal{K}$ is finitely axiomatizable if $\mathcal{K}$ satisfies SD($\wedge$) + $\mathsf{H}(\mathcal{K}) = \mathcal{K}$. M. Maróti, R. McKenzie [14] obtained a somewhat complicated result (see their Theorem 7.1) that extends both theorems of D. Pigozzi and R. Willard. Our Theorem 7 below extends the theorems of D. Pigozzi and R. Willard, and is a cleaner, more refined version of the result of M. Maróti and R. McKenzie.

Let $\mathcal{K}$ and $\mathcal{L}$ be locally finite quasivarieties of finite signature such that $\mathcal{K}$ is properly included in $\mathcal{L}$. A finite algebra $\mathbf{A}$ is said to be $(\mathcal{K}, \mathcal{L})$-*minimal* if $\mathbf{A} \notin \mathcal{K}$, $\mathbf{A} \in \mathcal{L}$, and every proper subalgebra of $\mathbf{A}$ belongs to $\mathcal{K}$. For a $(\mathcal{K}, \mathcal{L})$-minimal algebra $\mathbf{A}$, we let $\Theta_{\mathcal{K}}^{\mathbf{A}}$ denote the smallest congruence on $\mathbf{A}$ such that $\mathbf{A}/\Theta_{\mathcal{K}}^{\mathbf{A}} \in \mathcal{K}$.

Obviously, the relation of isomorphism partitions the family of all $(\mathcal{K}, \mathcal{L})$-minimal algebras. Let $\mathcal{S}$ be a selector set of that partition.

For $\mathbf{A} \in \mathcal{S}$, let $a \mapsto x_a$ be a fixed one-to-one assignment of individual variables to the elements of $\mathbf{A}$. Let $D(\mathbf{A})$ denote the collection of all equations of the form $f(x_{a_1}, \ldots, x_{a_n}) = x_{f(a_1, \ldots, a_n)}$, where f is an n-ary operation symbol of the signature of $\mathbf{A}$; and $a_1, \ldots, a_n$ are elements of $\mathbf{A}$. As the signature is finite and $\mathbf{A}$ is finite, $D(\mathbf{A})$ is finite. We set

$$\Sigma_{\mathbf{A}} = \{\bigwedge D(\mathbf{A}) \to x_a \approx x_b : \quad a \neq b \text{ and } (a, b) \in \Theta_{\mathcal{K}}^A\}.$$

For a set of quasi-identities Γ, we let $\mathrm{Mod}_{\mathcal{L}} \Gamma$ denote the set of algebras of $\mathcal{L}$ in which the quasi-identities of Γ are universally valid.

Proposition 5. $\mathcal{K} = \mathrm{Mod}_{\mathcal{L}} \left(\bigcup \{\Sigma_{\mathbf{A}} : \mathbf{A} \in S\} \right).$

Proof. $\subseteq$: Let $\mathbf{B} \in \mathcal{K}$ and $\mathbf{A} \in \mathcal{S}$. Let $(a, b) \in \Theta_{\mathcal{K}}^{\mathbf{A}}$ be such that $a \neq b$. Let $x_a \mapsto v(x_a)$ be an assignment of values in $\mathbf{B}$ of $\{x_a : a \in A\}$ under which the equations of $D(\mathbf{A})$ are satisfied. Notice that the map $a \mapsto v(x_a)$, denoted by φ, is a homomorphism from $\mathbf{A}$ to $\mathbf{B}$. As $\mathbf{B} \in \mathcal{K}$, $\mathbf{A}/\ker(\varphi) \in \mathcal{K}$. So

$\Theta_{\mathcal{K}}^{\mathbf{A}} \leq \ker(\varphi)$. This gives $\varphi(a) = \varphi(b)$. Thus $v(x_a) = v(x_b)$, proving that $\mathbf{B} \models \bigwedge D(A) \to x_a = x_b$, i.e. $\mathbf{B} \in \mathrm{Mod}_{\mathcal{L}}\left(\bigcup\{\Sigma_{\mathbf{A}} : \mathbf{A} \in S\}\right)$.

$\supseteq$: Let $\mathbf{B} \in \mathcal{L}$ be finite and $\mathbf{B} \notin \mathcal{K}$. Then there is $\mathbf{A} \in \mathcal{S}$ that is isomorphic to a subalgebra of $\mathbf{B}$. As $\mathbf{A} \not\models \Sigma_{\mathbf{A}}$, $\mathbf{B} \not\models \Sigma_{\mathbf{A}}$. Thus $\mathbf{B} \notin \mathrm{Mod}_{\mathcal{L}}\left(\bigcup\{\Sigma_{\mathbf{A}} : \mathbf{A} \in S\}\right)$. $\square$

Theorem 6. *Suppose that $\mathcal{K} \subseteq \mathcal{L} \subseteq \mathcal{L}'$ are quasivarieties of finite signature such that $\mathcal{K}$ satisfies WEP and $\mathcal{L}'$ is finitely generated. Then $\mathcal{K}$ is finitely axiomatizable relative to $\mathcal{L}$.*

Proof. Let n be an integer not less than the cardinality of any of the generators of $\mathcal{L}'$. Let m be the maximum size of an n-generated subalgebra of any algebra in $\mathcal{L}'$. We may assume that $\mathcal{K} \neq \mathcal{L}$. We claim that any finite algebra in $\mathcal{L} \setminus \mathcal{K}$ has a subalgebra of at most m elements that lies outside of $\mathcal{K}$. This will mean that the selector $\mathcal{S}$ as defined above is finite. This, in turn, by Proposition 5, will complete the proof.

To see the claim, we assume that $\mathbf{A}$ is a finite algebra in $\mathcal{L} \setminus \mathcal{K}$. For $X, Y \subseteq \mathrm{Con}\,\mathbf{A}$, we define $X \preceq Y$ iff for every $\alpha \in Y$ there exists $\beta \in X$ such that $\beta \leq \alpha$. Notice that $\preceq$ restricted to antichains in $\mathrm{Con}\,\mathbf{A}$ is a partial order.

Let X be an antichain in $\mathrm{Con}\,\mathbf{A}$ whose intersection is 0_A and such that for each $\alpha \in X$, the quotient algebra $\mathbf{A}/\alpha$ has at most n elements. Notice that such X exists for $\mathbf{A} \in \mathcal{L}$ since n bounds the cardinality of the generators of $\mathcal{L}'$. Now, among all antichains Y in $\mathrm{Con}\,\mathbf{A}$ with $X \preceq Y$ and with the intersection of Y equal to 0_A, we choose one, say Z, that is $\preceq$-maximal. It follows from the choice of Z that each member of Z is a meet-irreducible element of $\mathrm{Con}\,\mathbf{A}$. Since $\mathbf{A}$ is outside of $\mathcal{K}$, we can choose γ in Z so that $\mathbf{A}/\gamma \notin \mathcal{K}$. Let δ be the unique cover of γ in $\mathrm{Con}\,\mathbf{A}$. Since, by the choice, Z is $\preceq$-maximal, there is a pair (a,b) in $A \times A$ such that $(a,b) \in \delta \wedge \bigwedge(Z \setminus \{\gamma\})$ and $a \neq b$. Choose $a_0, \ldots, a_{k-1}$ in A to be a selector set for all the γ-equivalence classes in A with $a_0 = a$ and $a_1 = b$. (Note that $2 \leq k$ since $(a,b) \notin \gamma$.) As $X \preceq Z$, it follows that $k \leq n$.

Let $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ generated by $a_0, \ldots, a_{k-1}$. As $k \leq n$, $\mathbf{B}$ has at most m elements. Let γ' and σ denote the restrictions of γ and $\bigwedge(Z \setminus \{\gamma\})$ to the algebra $\mathbf{B}$, respectively. Obviously, $\gamma' \wedge \sigma = 0_B$. We now want to show that $\mathbf{B} \notin \mathcal{K}$. To this effect suppose otherwise that $\mathbf{B} \in \mathcal{K}$. Then, by WEP of $\mathcal{K}$, there are $\mathcal{K}$-congruences $\gamma'_{\mathcal{K}}$ and $\sigma_{\mathcal{K}}$ on $\mathbf{B}$ that extend γ' and σ, respectively, and are such that $\gamma'_{\mathcal{K}} \wedge \sigma_{\mathcal{K}} = 0_B$. As $a, b \in B$ and $(a,b) \in \bigwedge(Z \setminus \{\gamma\})$, it follows that $(a,b) \in \sigma \leq \sigma_{\mathcal{K}}$. This implies $(a,b) \notin \gamma'_{\mathcal{K}}$.

Since $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ generated by a selector set for all the γ-equivalence classes in $\mathbf{A}$, it follows that $\mathbf{B}/\gamma'$ is isomorphic to $\mathbf{A}/\gamma$. Thus γ' has the unique cover in $\mathrm{Con}\,\mathbf{B}$ to which (a,b) belongs; recall that $a_0 = a$ and $a_1 = b$. This implies that $\gamma'_{\mathcal{K}} = \gamma'$ for otherwise $\gamma'_{\mathcal{K}}$ would be above this unique cover and as a consequence (a,b) would belong to $\gamma'_{\mathcal{K}}$. But $\mathbf{A}/\gamma \notin \mathcal{K}$ and so $\mathbf{B}/\gamma' \notin \mathcal{K}$. Hence $\mathbf{B}/\gamma'_{\mathcal{K}} \notin \mathcal{K}$, a contradiction. Thus $\mathbf{B} \notin \mathcal{K}$. $\square$

We remark that in place of assuming $\mathcal{K}$ satisfies WEP in Theorem 6 it would suffice to assume, where $\mathcal{L}'$ is generated by algebras of at most n elements, that for every n-generated algebra $\mathbf{A} \in \mathcal{K}$ and for all $\alpha, \beta \in\in$ Con $\mathbf{A}$, if $\alpha \wedge \beta = 0_A$ then $\overline{\alpha}^{\mathcal{K}} \wedge \overline{\beta}^{\mathcal{K}} = 0_A$. (An examination of the proof reveals that this is all we used.)

In view of the fact that $\mathrm{RSD}(\wedge) \Leftrightarrow \mathrm{PCC} + \mathrm{WEP}$ for quasivarieties (Theorem 1 (4)), our next theorem is clearly identical with Corollary 6.4 in M. Maróti, R. McKenzie [14]. Those authors, however, did not know this characterization of $\mathrm{RSD}(\wedge)$, and we wish to present a somewhat cleaner proof of the result than they gave. Nevertheless, we shall have to use without proof one of their results.

Theorem 7. *Every finitely generated quasivariety of finite signature satisfying $\mathrm{RSD}(\wedge)$ is finitely axiomatizable.*

Proof. Let $\mathcal{K}$ be a finitely generated quasivariety of finite signature, satisfying $\mathrm{RSD}(\wedge)$. By Theorem 1 (4) above, $\mathcal{K}$ satisfies PCC and WEP. Now, Corollary 5.4 in M. Maróti, R. McKenzie [14] and its proof establishes that $\mathcal{K} \subseteq \mathcal{L} \subseteq \mathcal{L}'$ for a certain finitely axiomatizable quasivariety $\mathcal{L}$ and finitely generated quasivariety $\mathcal{L}'$. Then it follows from Theorem 6 that $\mathcal{K}$ is finitely axiomatizable. $\square$

As $\mathrm{RSD}(\wedge) \Leftrightarrow \mathrm{PCC} + \mathrm{WEP}$, one may ask whether Theorem 7 remains true if $\mathrm{RSD}(\wedge)$ is replaced in it by PCC or by WEP. The answer is negative in the first case (see W. Dziobiak [7], J. Ježek, M. Maróti, R. McKenzie [10], and R. McKenzie [18]) but it is unknown to us in the second case.

Theorem 8. *If $\mathcal{K}$ is a finitely generated quasivariety of finite signature which is RCM and $\mathsf{H}(\mathcal{K})$ is CM and residually small, then $\mathcal{K}$ is finitely axiomatizable.*

Proof. By Theorem 1(6), $\mathcal{K} \models \mathrm{WEP}$. The result now follows from Theorem 6, R. McKenzie's theorem (see introduction), and a result of R. Freese, R. McKenzie [8] which says that every finitely generated and residually small CM variety of finite signature is of the form $\mathcal{L} = \mathsf{SP}(\mathcal{M})$ for some finite set $\mathcal{M}$ of finite algebras. $\square$

Corollary 9. *Let $\mathcal{M}$ be a finite set of finite algebras of finite signature from a CM variety. If $\mathsf{SP}(\mathsf{HS}(\mathcal{M}))$ is RCM, then $\mathsf{SP}(\mathsf{HS}(\mathcal{M}))$ is finitely axiomatizable.*

Proof. Let $\mathbf{B} \in \mathsf{HS}(\mathcal{M})$ be subdirectly irreducible with monolith θ. Let ψ be a congruence on $\mathbf{B}$ such that the commutator $[\theta, \psi]$ calculated in the variety generated by $\mathcal{M}$ is equal to 0_B. As $\mathsf{SP}(\mathsf{HS}(\mathcal{M}))$ is RCM and $0_B = [\theta, \psi] = [\theta \wedge \psi, \psi]$, it follows from the note following Theorem 3.1 of K. Kearnes and R. McKenzie [11] that $[\psi, \psi] = 0_B$. Thus any subdirectly irreducible algebra $\mathbf{B}$ from $\mathsf{HS}(\mathcal{M})$ satisfies, where θ is the monolith of $\mathbf{B}$: If $\psi \in \mathrm{Con}\,\mathbf{B}$ and $[\theta, \psi] = 0_B$, then $[\psi, \psi] = 0_B$. This, by a result of R. Freese and R.

McKenzie [8], gives that $\mathsf{HSP}(\mathcal{M})$ is residually small. Thus, by Theorem 8, $\mathsf{SP}(\mathsf{HS}(\mathcal{M}))$ is finitely axiomatizable.   □

In the context of Theorem 8, it is worthwhile to mention a result of R. McKenzie [15] which says that if $\mathcal{V}$ is a locally finite variety which is residually finite and which satisfies some nontrivial congruence identity, then $\mathcal{V}$ is congruence modular.

5. Examples

This section contains four examples each of which provides a negative answer on some questions emerging naturally in the context of our work.

Example 10. This is an example of a finitely generated quasivariety $\mathcal{K}$ that satisfies $\mathrm{RSD}(\wedge) + \neg\mathrm{EP}$. We define a five-element algebra $\mathbf{A}$ and a two-element algebra $\mathbf{B} = \mathbf{A}/\gamma$ for a certain congruence γ of $\mathbf{A}$ and take $\mathcal{K} = \mathsf{SP}(\{\mathbf{A}, \mathbf{B}\})$. We put

$$\mathbf{A} = \langle \{0, a, b, c, 1\}, \wedge, \bar{0}, \bar{a}, \bar{b}, \bar{c}, \bar{1}, f_0, f_1 \rangle$$

so that $\langle \{0, a, b, c, 1\}, \wedge \rangle$ is a semilattice with least and largest elements $0, 1$ and with $a \wedge b = c$, the five elements of $\mathbf{A}$ are denoted by constants in the signature of $\mathbf{A}$, and f_0, f_1 are unary operations defined by $f_0(b) = f_0(1) = c$ and $f_0(x) = 0$ for $x \in \{0, a, c\}$, $f_1(a) = f_1(1) = c$ and $f_1(x) = 0$ for $x \in \{0, b, c\}$. This algebra is subdirectly irreducible and its congruence lattice is an eight-element lattice. Its least non-zero congruence is the equivalence relation μ that has $\{0, c\}$ as its only non-singleton equivalence class. Another congruence is γ that has $\{0, a, b, c\}$ and $\{1\}$ as equivalence classes. We take $\mathbf{B} = \mathbf{A}/\gamma$, and $\mathcal{K} = \mathsf{SP}(\{\mathbf{A}, \mathbf{B}\})$.

For any two distinct elements e, e' of A, and any $f, f' \in A \setminus \{1\}$, both $\mathbf{A}$ and $\mathbf{B}$ satisfy the quasi-equation $\bar{e} = \bar{e'} \rightarrow \bar{f} = \bar{f'}$. Hence the lattice of $\mathcal{K}$-congruences of $\mathbf{A}$ consists of $0_A, \gamma$, and 1_A. Among the other congruences of $\mathbf{A}$ are β_0 with equivalence classes $\{0, c, a\}, \{b, 1\}$ and β_1 with equivalence classes $\{0, c, b\}, \{a, 1\}$. We have $\beta_0 \wedge \beta_1 = \mu$ while $\overline{\beta_0}^{\mathcal{K}} = 1_A = \overline{\beta_1}^{\mathcal{K}}$ and $\overline{\mu}^{\mathcal{K}} = \gamma$. Thus $\mathcal{K}$ does not have the EP.

Since members of $\mathcal{K}$ are expansions of semilattices, $\mathcal{K}$ does have the PCC, even the $\mathrm{SD}(\wedge)$. To be able to conclude that $\mathcal{K} \models \mathrm{RSD}(\wedge)$, we must show that $\mathcal{K} \models \mathrm{WEP}$.

To see this, let $\mathbf{C}$ be any finite member of $\mathcal{K}$. We may assume that $\mathbf{C}$ is a subalgebra of $\mathbf{A}^{I_0} \times \mathbf{B}^{I_1}$ for finite, pairwise disjoint sets I_0, I_1 and not isomorphic to a subalgebra of $\mathbf{A}^{I_0'} \times \mathbf{B}^{I_1'}$ for any $I_0' \subseteq I_0, I_1' \subseteq I_1$ with $I_0' \neq I_0$ or $I_1' \neq I_1$.

It will suffice to prove the following: Let $0_C \prec \alpha$ be any atom in the congruence lattice of $\mathbf{C}$ and let β be the pseudo-complement of α—the largest congruence disjoint from α; then β is a $\mathcal{K}$-congruence of $\mathbf{C}$. For that purpose, we may choose $p, q \in C$ so that $(p, q) \in \alpha$, $q < p$ (in the semilattice order), and for all $p', q' \in C$ with $q' \leq p' < p$ and $(p', q') \in \alpha$ we have $p' = q'$.

<u>Case 0</u>: There is $i_0 \in I_0$ with $p(i_0) \neq q(i_0)$. If $p(i_0) \in \{a, b, 1\}$, then one of the pairs $(f_0(p), f_0(q)) = (p', q')$ or $(f_1(p), f_1(q)) = (p', q')$ has $p' > q'$. Since in both cases, $p' < p$, this is impossible. Thus $p(i_0) \in \{0, c\}$, forcing $(p(i_0), q(i_0)) = (c, 0)$. Now $(p \wedge \bar{c}, q \wedge \bar{c}) \in \alpha$, and it follows that $q < p \leq \bar{c}$.

Now let η_{i_0} denote the kernel of the projection of $\mathbf{C}$ onto $\mathbf{A}$ at the i_0th coordinate (the projection is onto since we have the constants in the signature), and let η'_{i_0} denote the kernel of the projection of $\mathbf{C}$ into $\mathbf{A}^{I_0 \setminus \{i_0\}} \times \mathbf{B}^{I_1}$. By the minimality of our subdirect representation of $\mathbf{C}$, there must be a pair $(r, s) \in \eta'_{i_0}$, $r \neq s$. We can assume that $s < r$. As above, we can also assume that $r \leq \bar{c}^{\mathbf{C}}$. Thus $(r(i_0), s(i_0)) = (c, 0)$. Now it is easy to verify that $p \geq r \wedge p > s \wedge p \geq r \wedge q$. Thus $(r \wedge p, s \wedge p) \in \alpha$. (By minimality of p, then $p = r \wedge p$.) The pair $(r \wedge p, s \wedge p)$ of functions, which differ only at i_0, must generate α; hence $\alpha \leq \eta'_{i_0}$. This implies that $\alpha \wedge \eta_{i_0} = 0_C$, and so $\beta \geq \eta_{i_0}$. Now $\mathbf{C}/\eta_{i_0} \cong \mathbf{A}$ and so η_{i_0} has a unique cover $\mu_{i_0} \in \text{Con}\,\mathbf{C}$, i.e. $(x, y) \in \mu_{i_0}$ iff $(x, y) \in C^2$ and $(x(i_0), y(i_0)) \in \mu$. Clearly, $(r \wedge p, s \wedge p) \in \mu_{i_0}$, hence $\beta \not\geq \mu_{i_0}$. So we conclude that $\beta = \eta_{i_0}$. But this implies that β is a $\mathcal{K}$-congruence of $\mathbf{C}$, as we wanted to prove.

Finally, we consider

<u>Case 1</u>: Where η_{I_0} is the kernel of the projection of $\mathbf{C}$ into $\mathbf{A}^{I_0}$ and η_{I_1} is the kernel of the projection of $\mathbf{C}$ into $\mathbf{B}^{I_1}$, we have that $\alpha \leq \eta_{I_0}$. In this case, $\alpha \wedge \eta_{I_1} = 0_C$, implying that $\beta \geq \eta_{I_1}$. But since $\mathbf{C}/\eta_{I_1}$ is embeddable into $\mathbf{B}^{I_1}$, it is easy to see that every congruence of $\mathbf{C}$ containing η_{I_1} is a $\mathcal{K}$-congruence. Thus β is a $\mathcal{K}$-congruence, also in Case 1. $\square$

Example 11. This is an example of a finitely generated quasivariety that satisfies $\text{SD}(\wedge)$ and not WEP, and hence PCC and not $\text{RSD}(\wedge)$. We remark that the examples of finitely generated quasivarieties of lattices that are not finitely axiomatizable supplied by, for example, V.P. Belkin [3], satisfy PCC and, as a consequence of our Theorem 1 and Theorem 7, cannot satisfy WEP. The example we now present is, however, much simpler to demonstrate.

We use a four-element algebra $\mathbf{A} = \langle \{0, a, b, 1\}, \wedge, \bar{0}, \bar{a}, \bar{b}, \bar{1} \rangle$. Here, $\langle A, \wedge \rangle$ is the four element meet-semilattice with two atoms a, b and the other operations are the constants. This algebra has the congruence γ with classes $\{0, a, b\}, \{1\}$ and we put $\mathbf{B} = \mathbf{A}/\gamma$ and $\mathcal{K} = \text{SP}(\{\mathbf{A}, \mathbf{B}\})$. $\mathbf{A}$ also has congruences β_0 with classes $\{0, a\}, \{1, b\}$ and β_1 with classes $\{0, b\}, \{1, a\}$. By an obvious argument very similar to the one we used in the last example, one shows that $\overline{\beta_0}^{\mathcal{K}} = 1_A = \overline{\beta_1}^{\mathcal{K}}$. However $\beta_0 \wedge \beta_1 = 0_A$, and this shows that $\mathcal{K}$ fails to have WEP. This quasivariety does have $\text{SD}(\wedge)$, due to the presence of the semilattice operation. $\square$

Example 12. This is an example of a finitely generated quasivariety that satisfies $\text{RCD} + \text{CM} + \neg \text{CD}$. We put $\mathcal{K} = \text{SP}(\mathbf{A})$ where $\mathbf{A}$ is a six-element algebra which we now define.

$$\mathbf{A} = \langle \{0, 1, 2, a, b, c\}, m, \wedge, \bar{0}, \bar{1}, \bar{2}, \bar{a}, \bar{b}, \bar{c}, F_0, F_a, F_b, F_c \rangle$$

has six constants, a ternary operation m, binary operation $\wedge$, and four unary operations F_r where $r \in U = \{0, a, b, c\}$. We define F_r so that $F_r(r) = 2$ and $F_r(x) = 1$ for all $x \in A \setminus \{r\}$. Let $D = \{1, 2\}$ and let $\langle U, + \rangle$ and $\langle D, + \rangle$ be Boolean groups with identity elements 0, 1 respectively. We define $m(x, y, z)$ so that $m(x_0, x_1, x_2) = x_0 + x_1 + x_2$ if $\{x_0, x_1, x_2\}$ is a subset of U, or of D; and otherwise, if $|\{i : x_i \in U\}| = 1$ and $\{p\} = \{x_0, x_1, x_2\} \cap U$, or if $|\{i : x_i \in D\}| = 1$ and $\{p\} = \{x_0, x_1, x_2\} \cap D$ then we put $m(x_0, x_1, x_2) = p$. Thus m is fully defined as an operation on A. We define $x \wedge y$ to be the minimum element in $\{x, y\} \cap D$ if $\{x, y\} \cap D$ is non-empty, and otherwise, $x \wedge y = 2$.

Now m is a Maltsev operation, i.e., it satisfies the equations $m(x, y, y) = x = m(y, y, x)$ on A. Thus $\mathcal{K}$ satisfies CM. The equivalence relation μ with one non-trivial block $\{1, 2\}$ is a congruence of $\mathbf{A}$, in fact, the smallest non-zero congruence of $\mathbf{A}$. Besides 0_A and 1_A and μ, this algebra has precisely four further congruences; they are the equivalence relations $\mu \cup \beta$ where β is any non-zero congruence on $\langle U, + \rangle$. The congruence lattice of $\langle U, + \rangle$ is in this way embedded as an interval in $\mathbf{Con\,A}$. This shows that $\mathcal{K}$ does not satisfy CD.

Finally, we must demonstrate that $\mathcal{K} \models \mathrm{RCD}$. It will suffice to show that every finite algebra in $\mathcal{K}$ has a distributive lattice of $\mathcal{K}$-relative congruences. So let $\mathbf{B}$ be a finite algebra in $\mathcal{K}$. We can assume that $\mathbf{B} \subseteq \mathbf{A}^n$ where n is a positive integer, and that for all $1 \leq m < n$, $\mathbf{B}$ is not embeddable into $\mathbf{A}^m$. For $0 \leq i < n$ let 0_i be the kernel of the projection homomorphism mapping of $\mathbf{B}$ to $\mathbf{A}$ at the ith coordinate. Because n has been choosen to be minimal, the set $P = \{0_0, \ldots, 0_{n-1}\}$ is an n-element subset of $\mathrm{Con}_{\mathcal{K}}\mathbf{B}$, and moreover, the set C of all intersections of subsets of P is a 2^n-element subset of $\mathrm{Con}_{\mathcal{K}}\mathbf{B}$. Actually, if $\{\phi\} \cup \Gamma \subseteq P$ then $\phi \geq \bigwedge \Gamma$ iff $\phi \in \Gamma$. Our objective is to prove that $C = \mathrm{Con}_{\mathcal{K}}\mathbf{B}$. If this is true, then it easily follows that the map $\Gamma \mapsto \bigwedge \Gamma$ is an isomorphism of the Boolean lattice of all subsets of P onto the lattice $\mathbf{Con}_{\mathcal{K}}\mathbf{B}$. In this way, $\mathbf{Con}_{\mathcal{K}}\mathbf{B}$ will be shown to be distributive.

To accomplish this objective, suppose that θ is an arbitrary member of $\mathrm{Con}_{\mathcal{K}}\mathbf{B}$. Set T equal to the set of all $\eta \in P$ such that $\theta \leq \eta$, and put $\theta' = \bigwedge T$. So $\theta \leq \theta'$ and we have to prove that $\theta = \theta'$.

<u>Claim 1:</u> $D^n \subseteq B$.

To prove this, choose any $0 \leq i < n$. Now since $0_i \not\geq \bigwedge(P \setminus \{0_i\})$, there are $f, g \in B$ with $f(j) = g(j)$ exactly for $j \neq i$, $0 \leq j < n$. If $\{f(i), g(i)\} \cap U \neq \emptyset$, say $f(i) = r \in U$, then where $f' = F_r(f)$, $g' = F_r(g)$, we have $\{f', g'\} \subseteq B \cap D^n$, $(f'(i), g(i)) = (2, 1)$, and $f'(j) = g'(j)$ for all $j \neq i$. Then put $f'' = m(f', g', \bar{1})$ and notice that $f'' \in D^n \cap B$ is the function $[1, 2]_i$ that takes value 2 at i and 1 elsewhere. On the other hand, if $\{f(i), g(i)\} \subseteq D$ then put $f'' = m(f \wedge \bar{2}, g \wedge \bar{2}, \bar{1})$ and again, $f'' = [1, 2]_i \in D^n \cap B$. We have shown that B contains all the functions $[1, 2]_i$, $0 \leq i < n$. Now if $f, g \in D^n$ and $f \wedge g = \bar{1}$, then $m(\bar{1}, f, g) = h$ is $f \vee g$—i.e., $h(i) = 2$ iff $f(i) = 2$ or $g(i) = 2$ for all $0 \leq i < n$. It should now be clear that $D^n \subseteq B$.

<u>Claim 2:</u> If $\eta = 0_i \in P$, then $\theta \not\leq \eta$ iff $(\bar{1}, [1,2]_i) \in \theta$ and $(\bar{2}, [2,1]_i) \in \theta$.

Indeed, suppose that $\theta \not\leq 0_i$. Choose $(f,g) \in \theta$ with $f(i) \neq g(i)$. Then as above, by either replacing (f,g) by $(F_r(f), F_r(g))$ for some $r \in U \cap \{f(i), g(i)\}$, or by replacing (f,g) by $(f \wedge \bar{2}, g \wedge \bar{2})$, we get $(f', g') \in \theta \cap (D^n \times D^n)$, $f'(i) \neq g'(i)$. Now $\{f' \wedge [1,2]_i, g' \wedge [1,2]_i\} = \{\bar{1}, [1,2]_i\}$; and so we have $(\bar{1}, [1,2]_i) \in \theta$. Then also

$$((\bar{2}, [2,1]_i) = (m(\bar{1}, \bar{2}, \bar{1}), m(\bar{1}, \bar{2}, [1,2]_i)) \in \theta \,,$$

as required.

<u>Claim 3:</u> $\theta' \cap (D^n \times D^n) \subseteq \theta$.

To prove this, let

$$S = \{0, \ldots, n-1\} \setminus T = \{i_0, \ldots, i_{k-1}\}, \ |S| = k \,.$$

Then $(\bar{2}, [2,1]_S) \in \theta$. This is clear if $k = 0$ for then $[2,1]_S = \bar{2}$. It is also clear if $k = 1$. If $k > 1$ then

$$
\begin{aligned}
[2,1]_S &= m(m(\cdots m([2,1]_{i_0}, \bar{2}, [2,1]_{i_1}), \ldots,), \bar{2}, [2,1]_{i_{k-1}}) \\
&\equiv m(m(\bar{2}, \bar{2}, \bar{2}), \ldots,), \bar{2}, \bar{2}) \pmod{\theta} \\
&= \bar{2} \,.
\end{aligned}
$$

Now let $(f,g) \in \theta'$, $\{f, g\} \subseteq D^n$. Since f and g agree on T, then $f \wedge [2,1]_S = g \wedge [2,1]_S$. Also, $f = f \wedge \bar{2}$ and $g = g \wedge \bar{2}$. From $([2,1]_S, \bar{2}) \in \theta$, we conclude that $(f,g) \in \theta$.

Finally, to finish this proof that $\theta = \theta'$, let us suppose that $(f,g) \in \theta' \setminus \theta$. Since $\theta \in \mathrm{Con}_\mathcal{K} \mathbf{B}$ by assumption, then there is a homomorphism $\pi : \mathbf{B} \to \mathbf{A}$ with $\pi(f) \neq \pi(g)$ and $\theta \subseteq \ker(\pi)$. Let $r \in U$ be arbitrary. Now $(F_r(f), F_r(g)) \in \theta' \cap (D^n \times D^n)$ hence by Claim 3, $(F_r(f), F_r(g)) \in \theta$ and so $\pi(F_r(f)) = \pi(F_r(g))$. This means that $F_r(\pi(f)) = F_r(\pi(g))$ for all $r \in U$; and that implies that $\{\pi(f), \pi(g)\} \subseteq D$ since $\pi(f) \neq \pi(g)$. Similarly, we have that $(f \wedge \bar{2}, g \wedge \bar{2}) \in \theta$ and $\pi(f) \wedge \bar{2} = \pi(g) \wedge \bar{2}$. Since $\{\pi(f), \pi(g)\} \subseteq D$, then

$$\pi(f) = \pi(f) \wedge \bar{2} = \pi(g) \wedge \bar{2} = \pi(g) \,.$$

This contradiction completes our proof that $\theta = \theta'$. $\qquad \square$

Example 13. This is an example, due to R. Willard, of a finitely generated quasivariety $\mathcal{K}$ with RCM and such that $\mathsf{H}(\mathcal{K})$ is CM but is not residually small. We define a finite algebra $\mathbf{A}$ with universe $A := \mathbb{Z}_4 \times \mathbb{Z}_2$ as follows. First define $\lambda_i : \mathbb{Z}_4 \to \mathbb{Z}_2$ for $i < 2$ by

x	$\lambda_0(x)$	$\lambda_1(x)$
0	0	0
1	0	1
2	1	0
3	1	1

Then define

$$
\begin{aligned}
f_i(\langle x, r \rangle) &= \langle 0, \lambda_i(x) \rangle, \quad i = 0, 1 \\
f_2(\langle x, r \rangle) &= \langle 0, r \rangle \\
\langle x, r \rangle + \langle y, s \rangle &= \langle x + y, \, r + s \rangle \\
\langle x, r \rangle \cdot \langle y, s \rangle &= \langle 2xy, rs \rangle
\end{aligned}
$$

Observe that $\langle A; +, \cdot \rangle$ is a ring (without identity) isomorphic to $2\mathbb{Z}_8 \times \mathbb{Z}_2$. Let $A_0 = \{0\} \times \mathbb{Z}_2 \subseteq A$ and write $\mathbf{0}$ for $\langle 0, 0 \rangle$ and $\mathbf{1}$ for $\langle 0, 1 \rangle$. Then we can write $A_0 = \{\mathbf{0}, \mathbf{1}\} = \mathbf{2}$, which is convenient since A_0 is closed under $+$, $\cdot$ and with respect to these operations is a 2-element boolean subring of $\langle A; +, \cdot \rangle$. We define

$$
\mathbf{A} = \langle A; +, \cdot, \underline{\mathbf{0}}, \underline{\mathbf{1}}, f_i \ (i < 3) \rangle
$$

and $\mathcal{K} = \mathsf{SP}(\mathbf{A})$.

Clearly, $\mathsf{H}(\mathcal{K})$ is congruence modular. We shall show that $\mathcal{K}$ has WEP and $\mathsf{H}(\mathcal{K})$ is residually large. We first show that $\mathbf{Con\,A}$ is a 4-element chain with congruences $0_A < \mu < \theta < 1_A$ where

$$
\begin{aligned}
\mu &= 0_{\mathbb{Z}_4} \times 1_{\mathbb{Z}_2} \\
\theta &= \gamma \times 1_{\mathbb{Z}_2}
\end{aligned}
$$

and γ is $\{0, 2\}^2 \cup \{1, 3\}^2$.

First check that μ is a congruence. Next observe that the presence of $+$ implies that $0_A \prec \mu$. Finally, the presence of f_0, f_1 guarantee that if $a, b \in A$ with $a \neq b$ then $\mathrm{Cg}_{\mathbf{A}}(a, b) \cap \mu \neq 0_A$, proving μ is the monolith of $\mathbf{A}$. Now $\mathbf{A}/\mu$ is term-equivalent to the 4-element ring $2\mathbb{Z}_8$, whose congruences are known, completing the picture of $\mathbf{Con\,A}$.

Since the ring $2\mathbb{Z}_8$ generates a residually large variety, so does $\mathbf{A}/\mu$. Thus $\mathsf{H}(\mathcal{K})$ is residually large.

It remains to prove WEP. It will suffice to show that each finite $\mathbf{B} \in \mathcal{K}$ has WEP. Choose finite $\mathbf{B} \in \mathcal{K}$ and represent it as $\mathbf{B} \leq \mathbf{A}^n$ with n minimum. For each $i < n$ let $\mu_i' = (0_A \times \cdots \times 0_A \times \mu \times 0_A \times \cdots \times 0_A)|_B$ where the lone copy of μ occurs in the i'th coordinate. Similarly let as usual $0_i = (1_A \times \cdots \times 1_A \times 0_A \times 1_A \times \cdots \times 1_A)|_B$ and $0_i' = (0_A \times \cdots \times 0_A \times 1_A \times 0_A \times \cdots \times 0_A)|_B$. Then $\mu_i, 0_i, 0_i' \in \mathrm{Con\,}\mathbf{B}$.

Put $B_0 = B \cap (A_0)^n$. Observe that $f_i(B) \subseteq B_0$ for $i < 3$, and that $\langle B_0; +, \cdot, \underline{\mathbf{0}}, \underline{\mathbf{1}} \rangle$ is a boolean subring of $\mathbf{2}^n$. For each $i < n$ let

$$
e_i = (\mathbf{0}, \ldots, \mathbf{1}, \ldots, \mathbf{0}) \in \mathbf{2}^n
$$

with the single $\mathbf{1}$ occurring in the i'th coordinate.

Claim 1: $e_i \in B_0$ for each $i < n$.

To prove the claim, Fix $i < n$. By minimality of n we have $a, b \in B$ with $[\![a \neq b]\!] = \{i\}$. There exists $j < 3$ such that $f_j(a_i) \neq f_j(b_i)$, so by replacing a, b with $a' := f_j(a)$, $b' := f_j(b)$ we may assume with no loss of generality that $a, b \in B_0$. Then $e_i = a + b \in B$ as required. Since B_0 is a boolean subring of $\mathbf{2}^n$, it follows as a corollary that $B_0 = \mathbf{2}^n$.

<u>Claim 2:</u> If $i < n$ and $c, d \in B$ and $c_i \neq d_i$, then $\mu_i' \leq \mathrm{Cg}_{\mathbf{B}}(c, d)$.

As in the proof of Claim 1, we have a unary term operation $f(x)$ such that $f(B) \subseteq B_0$ and $\{f(c)_i, f(d)_i\} = \{\mathbf{0}, \mathbf{1}\}$. The polynomial $x \mapsto x \cdot e_i$ maps $\{f(c), f(d)\}$ to $\{\underline{\mathbf{0}}, e_i\}$, proving $(\underline{\mathbf{0}}, e_i) \in \mathrm{Cg}_{\mathbf{B}}(c, d)$. Then for every $a \in B$, $(a, a + e_i) = (a + \underline{\mathbf{0}}, a + e_i) \in \mathrm{Cg}_{\mathbf{A}}(c, d)$, proving $\mu_i' \leq \mathrm{Cg}_{\mathbf{A}}(c, d)$.

Now to prove that $\mathbf{B}$ has WEP, we use Lemma 14 from the next section. Let α be a minimal congruence of $\mathbf{B}$ and let $\beta \in \mathrm{Con}\,\mathbf{B}$ be maximal with respect to satisfying $\alpha \wedge \beta = 0_B$. Then by Claim 2, $\alpha = \mu_i'$ for some $i < n$ and $\beta \leq 0_i$. Since $\mu_i' \wedge 0_i = 0_B$ we have $0_i \leq \beta$. Therefore $\beta = 0_i$, so $\beta \in \mathrm{Con}_{\mathcal{K}}(\mathbf{B})$ as required. $\qquad\square$

Notice that since every subalgebra $\mathbf{B}$ of $\mathbf{A}$ is subdirectly irreducible with non-Abelian monolith $\mathrm{Cg}_{\mathbf{B}}(\mathbf{0}, \mathbf{1})$, it follows by Theorem 2.3 and 4.1 of M. Maróti and R. McKenzie [14] that $\mathcal{K}$ has Willard terms. Hence, by Theorem 1(4), $\mathcal{K}$ is RSD($\wedge$) and thus, by Theorem 7, $\mathcal{K}$ is finitely axiomatizable.

6. A CHARACTERIZATION OF FINITE $\mathcal{F}$ FOR WHICH $\mathsf{SP}(\mathcal{F}) \models \mathrm{RSD}(\wedge)$

Let $\mathcal{F}$ be any finite set of finite algebras of the same signature. According to Theorem 4.1 in M. Maróti, R. McKenzie [14], $\mathsf{SP}(\mathcal{F})$ satisfies PCC if and only if every minimal congruence of every algebra in $\mathsf{S}(\mathcal{F})$ is non-Abelian. Our characterization of those $\mathcal{F}$ for which $\mathsf{SP}(\mathcal{F}) \models \mathrm{RSD}(\wedge)$, equivalently, $\mathsf{SP}(\mathcal{F}) \models \mathrm{PCC} + \mathrm{WEP}$, is a little more complicated. If a quasivariety $\mathcal{K}$ satisfies PCC then there are several ways to characterize WEP for $\mathcal{K}$. We use α^c to denote the pseudo-complement of a congruence α on some algebra in $\mathcal{K}$. Supposing that $\mathcal{K} \models \mathrm{PCC}$, these statements are equivalent to WEP for $\mathcal{K}$:

- For $\mathbf{A} \in \mathcal{K}$ and $\alpha, \beta \in \mathrm{Con}\,\mathbf{A}$ with $\alpha \wedge \beta = 0_A$, $\overline{\alpha}^{\mathcal{K}} \wedge \overline{\beta}^{\mathcal{K}} = 0_A$.
- For $\mathbf{A} \in \mathcal{K}$ and $\alpha, \beta \in \mathrm{Con}\,\mathbf{A}$ with $\alpha \wedge \beta = 0_A$, $\alpha \wedge \overline{\beta}^{\mathcal{K}} = 0_A$.
- For $\mathbf{A} \in \mathcal{K}$ and $\alpha \in \mathrm{Con}\,\mathbf{A}$, $\alpha^c \in \mathrm{Con}_{\mathcal{K}}\mathbf{A}$.

If $\mathcal{K}$ is locally finite but does not necessarily satisfy PCC, there is another way to characterize WEP.

Lemma 14. *Let $\mathcal{K}$ be a locally finite quasivariety.*

(1) *$\mathcal{K}$ satisfies WEP iff for every finite $\mathbf{A} \in \mathcal{K}$ and atom $\alpha \in \mathrm{Con}\,\mathbf{A}$, all maximal members of the set $\{\beta \in \mathrm{Con}\,\mathbf{A} : \beta \not\geq \alpha\}$ belong to $\mathrm{Con}_{\mathcal{K}}\mathbf{A}$.*

(2) *If $\mathcal{K}$ satisfies PCC then $\mathcal{K}$ satisfies WEP iff for every finite $\mathbf{A} \in \mathcal{K}$ and atom $\alpha \in \mathrm{Con}\,\mathbf{A}$, the pseudo-complement of α belongs to $\mathrm{Con}_{\mathcal{K}}\mathbf{A}$.*

Proof. Suppose that α is an atom in $\mathrm{Con}\,\mathbf{A}$ and $\beta \not\geq \alpha$ while for every congruence $\gamma > \beta$, we have $\gamma \geq \alpha$. Then β is strictly meet irreducible and its unique cover is $\beta \vee \alpha$. If $\mathcal{K}$ has the weak extension property, then $\overline{\beta}^{\mathcal{K}} \wedge \overline{\alpha}^{\mathcal{K}} = 0_A$, implying that $\overline{\beta}^{\mathcal{K}} = \beta$, so $\beta \in \mathrm{Con}_{\mathcal{K}}\mathbf{A}$. This shows the necessity of the condition in (1). To see that this condition is also sufficient, suppose that $\mathcal{K}$ does not have the weak extension property. Then in fact

there is some finite $\mathbf{A} \in \mathcal{K}$ and congruences α, β such that $\alpha \wedge \beta = 0_A$ and $\alpha \wedge \overline{\beta}^{\mathcal{K}} > 0_A$. Let α_0 be any atom (minimal congruence) below $\alpha \wedge \overline{\beta}^{\mathcal{K}}$. Now $\alpha_0 \wedge \beta = 0_A$. Let β' be any maximal member of the set of congruences τ with $\alpha_0 \not\leq \tau$ and $\beta \leq \tau$. We have to see that $\beta' \notin \mathrm{Con}_{\mathcal{K}}\mathbf{A}$. But if $\beta' \in \mathrm{Con}_{\mathcal{K}}\mathbf{A}$, then $\overline{\beta}^{\mathcal{K}} \leq \beta'$ implying that $\overline{\beta}^{\mathcal{K}} \not\geq \alpha_0$, which is not the case. This completes our argument for (1).

Now assume that $\mathcal{K}$ satisfies PCC. Then where $\mathbf{A} \in \mathcal{K}$ and α is an atom in $\mathrm{Con}\,\mathbf{A}$, there is a unique maximal member of the set of congruences not dominating α, namely α^c. Thus (2) can be seen as a specialization of (1) to this case. $\qquad\square$

If $\mathcal{K}$ is any quasivariety and if α is an atom in the congruence lattice of an algebra $\mathbf{A} \in \mathcal{K}$, and if the pseudocomplement α^c of α exists, then $\mathbf{A}/\alpha^c$ is a subdirectly irreducible algebra. Unless this subdirectly irreducible algebra belongs to $\mathcal{K}$, then $\mathcal{K}$ cannot have WEP. These observations imply, for instance that if $\mathcal{K}$ is locally finite and satisfies RSD($\wedge$) then it is generated by its finite subdirectly irreducible members.

Recall that when α is a non-Abelian atom in the congruence lattice of a finite algebra, the pseudo-complement of α does exist. (See D. Hobby, R. McKenzie [9].)

Lemma 15. *Assume that $\mathcal{K}$ is a quasivariety, $\mathcal{K} \models PCC$, $\{\mathbf{A}_0, \ldots, \mathbf{A}_k\}$ is a finite collection of finite algebras in $\mathcal{K}$, and $\mathbf{A} \leq \prod_i \mathbf{A}_i$ is a subdirect product. Let α be an atom in $\mathbf{Con}\,\mathbf{A}$ and $\beta = \alpha^c$. Choose i, $0 \leq i \leq k$, such that $\alpha \not\leq 0_i$. Then*

(i) *$0_i \leq \beta$; and*

(ii) *if $|\mathbf{A}_i| \leq n$ and the cardinality of the free algebra on n generators in $\mathcal{K}$ is m, then there are $\mathbf{B}_0, \ldots, \mathbf{B}_{m-1} \in \mathsf{S}(\{\mathbf{A}_0, \ldots, \mathbf{A}_k\})$ with $\mathbf{B}_0 = \mathbf{A}_i$, and there is $\mathbf{B} \leq \prod_j \mathbf{B}_j$ a subdirect product, and an atom α' in $\mathbf{Con}\,\mathbf{B}$, such that where $\beta' = \alpha'^c$, we have $\beta' \geq 0_0$ and $\mathbf{A}/\beta \cong \mathbf{B}/\beta' \in \mathsf{H}(\mathbf{B}_0) = \mathsf{H}(\mathbf{A}_i)$.*

Proof. Clearly, since $\alpha \not\leq 0_i$ then $\alpha \wedge 0_i = 0_A$, implying that $0_i \leq \alpha^c = \beta$. Choose a subset $\{a_0, \ldots, a_{n-1}\} \subseteq A$ which intersects all 0_i-classes and such that $a_0 \neq a_1$ and $(a_0, a_1) \in \alpha$. Let $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ generated by $\{a_0, \ldots, a_{n-1}\}$. Thus $|\mathbf{B}| \leq m$. Note that $(a_0, a_1) \notin 0_i$, else $\alpha = \mathrm{Cg}_{\mathbf{B}}(a_0, a_1) \leq 0_i$. It is then easy to see that there are $j_0 = i, j_1, \ldots, j_{m-1} \in \{0, \ldots, k\}$ such that $(\bigwedge_{0 \leq r < m} 0_{j_r})|_{\mathbf{B}} = 0_B$. Let $\beta' = \beta|_{\mathbf{B}}$ and $\alpha'' = \alpha|_{\mathbf{B}}$. Clearly, $\alpha'' \wedge \beta' = 0_B$, $\alpha'' > 0_B$, and since $\mathbf{B}$ projects onto $\mathbf{A}_i = \mathbf{A}_{j_0}$, every congruence strictly larger than β' contains α''. β' is the pseudo-complement of α'' in $\mathrm{Con}\,\mathbf{B}$. Let α' be any atom of $\mathrm{Con}\,\mathbf{B}$ lying below α''. Then $\beta' = \alpha'^c$ in $\mathrm{Con}\,\mathbf{B}$. Now let $\mathbf{B}_r \subseteq \mathbf{A}_{j_r}$ be the projection of $\mathbf{B}$ at the j_r coordinate. Thus $\mathbf{B}_0 = \mathbf{A}_i$. Replacing $\mathbf{B}$ by the subdirect product of $\mathbf{B}_0, \ldots, \mathbf{B}_{m-1}$ canonically isomorphic to it, and β', α' by the corresponding congruences, we have the desired conclusions. $\qquad\square$

Theorem 16. *Suppose that $\mathcal{K} = \mathsf{SP}(\mathcal{F})$ where $\mathcal{F}$ is a finite collection of finite algebras and $\mathsf{S}(\mathcal{F}) = \mathcal{F}$. Let $n = \max(|\mathbf{A}| : \mathbf{A} \in \mathcal{F})$ and let f be the cardinality of the free algebra on n generators in $\mathcal{K}$. Then $\mathcal{K} \models RSD(\wedge)$ iff the following are true.*

(i) *Every atom in the congruence lattice of any member of $\mathcal{F}$ is non-Abelian.*

(ii) *Let $\mathbf{A} \leq \prod_{0 \leq i < f} \mathbf{A}_i$ be a subdirect product with $\{\mathbf{A}_0, \ldots, \mathbf{A}_{f-1}\} \subseteq \mathcal{F}$. Let α be any atom in $Con\,\mathbf{A}$ and $\beta = \alpha^c$. Then $\mathbf{A}/\beta$ is isomorphic to a member of $\mathcal{F}$.*

Proof. The necessity of (i) follows from M. Maróti, R. McKenzie [14], Theorem 4.1. To see that (ii) is necessary, observe that by Lemma 14, $\mathbf{A}/\beta$ must belong to $\mathcal{K}$; but it is a subdirectly irreducible algebra and therefore, must actually be isomorphic to a subalgebra of some member of $\mathcal{F}$.

Now suppose that (i) and (ii) hold. Then we have that $\mathcal{K} \models$ PCC. To establish that $\mathcal{K} \models$ WEP, using Lemma 14 (2), let $\mathbf{A} \leq \prod_{0 \leq i \leq k} \mathbf{A}_i$ a subdirect product, with k a non-negative integer and $\mathbf{A}_i \in \mathcal{F}$. Let α be an atom in $\mathbf{Con\,A}$ and $\beta = \alpha^c$. Now it follows from Lemma 15, in conjunction with (ii), that $\mathbf{A}/\beta \in \mathcal{K}$. $\qquad\square$

It is obvious that the above theorem supplies an algorithm to effectively recognize whether $\mathsf{SP}(\mathcal{F})$ is $RSD(\wedge)$. It is not obvious if there may be some algorithm to solve this problem which lies low in the complexity hierarchy.

To conclude, we present three related problems that seem to us very interesting and very difficult.

Problem 17. Suppose that $\mathcal{K}$ is a finitely generated quasivariety satisfying CM + RCM. Is it finitely axiomatizable?

Problem 18. Suppose that $\mathcal{K}$ is a finitely generated quasivariety satisfying RCM. Is it finitely axiomatizable?

Problem 19. Suppose that $\mathcal{K}$ is a finitely generated quasivariety satisfying WEP. Is it finitely axiomatizable?

References

[1] K. Baker, *Primitive satisfaction and equational problems for lattices and other algebras,* Trans. Amer. Math. Soc. **190**(1974), 125-150.

[2] K. Baker, *Finite equational bases for finite algebras in a congruence distributive equational class,* Advances in Math. **24**(1977), 207-243.

[3] V. P. Belkin, *Quasi-identities of finite rings and lattices,* Algebra i Logika **17** (1978), 247–259.

[4] W.J. Blok, D. Pigozzi, *A finite basis theorem for quasivarieties,* Algebra Universalis **22**(1986), 1-13.

[5] J. Czelakowski, W. Dziobiak, *Congruence distributive quasivarieties whose finitely subdirectly irreducible members for a universal class,* Algebra Universalis **27**(1990), 128-149.

[6] W. Dziobiak, *Finitely generated congruence distributive quasivarieties of algebras,* Fundamenta Mathematicae **133** (1989), 47-57.

[7]   W. Dziobiak, *Quasivarieties of Sugihara semilattices with involution,* Algebra and Logic 39(2000), 26-36.

[8]   R. Freese, R. McKenzie, *Residually small varieties with modular congruence lattices,* Trans. Amer. Math. Soc. **264**(1981), 419-430.

[9]   D. Hobby, R. McKenzie, *The structure of finite algebras,* American Math. Soc., Contemporary Mathematics, vol. 76, 1988.

[10] J. Ježek, M. Maróti, R. McKenzie, *Quasi-equational theories of flat algebras,* Czech Math. J. **55**(2005), 665–675.

[11] K. Kearnes, R. McKenzie, *Commutator theory for relatively modular quasivarieties,* Trans. Amer. Math. Soc. **331**(1992), 465-502.

[12] R. Lyndon, *Identities in two-valued calculi,* Trans. Amer. Math. Soc. **71**(1951), 457-465.

[13] R. Lyndon, *Identities in finite algebras,* Proc. Amer. Math. Soc. **5**(1954), 8-9.

[14] M. Maróti, R. McKenzie, *Finite basis problems and results for quasivarieties,* Studia Logica **78**(2004), 293–320.

[15] R. McKenzie, *Finite forbidden lattices,* Universal algebra and lattice theory (Puebla, 1982), 176-205, Lecture Notes in Math. 1004, Springer, Berlin, 1983.

[16] R. McKenzie, *Finite equational bases for congruence modular varieties,* Algebra Universalis **24**(1987), 224–250.

[17] R. McKenzie, *Tarski's finite basis problem is undecidable,* International Journal of Algebra and Computation **6**(1996), 49–104.

[18] R. McKenzie, *A compatible flat algebra without finite quasi-equational basis,* preprint.

[19] A. Nurakunov, *Characterization of relatively distributive quasivarieties of algebras,* Algebra and Logic **29**(1990), 451-458.

[20] A. Nurakunov, *Quasi-identities of congruence-distributive quasivarieties of algebras,* Sibirian Math. J. **42**(2001), 108-118.

[21] D. Pigozzi, *Finite basis theorems for relatively congruence distributive quasivarieties,* Trans. Amer. Math. So. **310**(1988), 499-533.

[22] R. Willard, *A finite basis theorem for residually finite, congruence meet-semidistributive varieties,* Journal of Symbolic Logic **65**, 187-200.

[23] R. Willard, *Extending Baker's theorem,* Algebra Universalis **45**(2001), 335-344.

Department of Mathematics, University of Puerto Rico, Mayagüez Campus, Mayagüez PR 00681–9018, USA
   E-mail address: `w.dziobiak@gmail.com`

University of Szeged, Bolyai Institute, H6720 Szeged, Hungary
   E-mail address: `mmaroti@math.u-szeged.hu`

Department of Mathematics, Vanderbilt University, Nashville, TN 37235, USA
   E-mail address: `mckenzie@math.vanderbilt.edu`

Chui pr., 265a, Institute of Mathematics, National Academy of Science, Bishkek, 720071, Kyrghyz Republic
   E-mail address: `anvar@aknet.kg`